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THE INTERACTION OF NON-LINEAR WAVES IN A SLIGHTLY ANISOTROPIC VISCOELASTIC MEDIUM†

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Motions in the form of plane slightly non-linear quasi-transverse waves in a viscoelastic medium with small anisotropy are investigated. The problem of the interaction between two shock waves moving in opposite directions is considered.

THE PROBLEM of the interaction between two shock waves, one of which overtakes the other, if the state after the system of these two waves corresponds to a point from the region of non-uniqueness of the solution of the self-similar problem, was solved numerically in [1, 2].

To investigate the interaction between two shock waves moving in opposite directions one should, generally speaking, use the complete system of equations of the theory of elasticity (see, for example, [3]). However, in the case of slight non-linearity the problem can be simplified considerably and the necessary calculations can be carried out using a simplified system of equations [4].

We will assume that the oppositely propagating shock waves interact over a limited period of time τ. Hence, we can neglect the effect of the slight non-linearity and the weak anisotropy during the period that the oppositely travelling waves interact.

We will estimate the order of the error which occurs as a result of this assumption. Suppose the deformations which the medium possesses and those which arise after the waves have passed are small and are of the order of ε. Then, the terms in the equations of the complete system of equations of the theory of elasticity, which describe the slight non-linearity and the weak anisotropy, are of the order of $L^{-1}\varepsilon\chi$, $\chi = \max\{\varepsilon^2, g\}$ [4], where g is a parameter representing the anisotropy, and L is the characteristic scale of the length of the interacting waves. We will assume that the viscous terms are of the same order of magnitude.

For the oppositely travelling waves $\tau \sim L/C$, where C is the relative velocity of the waves. Integrating the non-linear terms along the characteristics of the corresponding waves, considered as linear, we obtain that the changes caused by the non-linear and viscous terms in a time τ, are of the order of $C^{-1}\varepsilon\chi$. The latter quantity can be neglected compared with ε, so that the waves after a time behave as linear, i.e. they do not change their form, and they then change only after a much longer time. The reason for these changes is the fact that each of the waves begins to move over a changed background after a time τ (the wave no longer corresponds to the background).

Hence, as result of the interaction between the two shock waves moving in opposite directions, two systems of waves are formed, travelling in opposite directions. To investigate the formation and evolution of each of these two systems of waves we will use the approximate equations [4] describing slightly non-linear quasi-transverse waves in a viscoelastic medium with small anisotropy, propagating only in one direction

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$$\frac{\partial u_\alpha}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial u_\alpha}-\frac{1}{2\sqrt{\mu\rho_0}}\tau_{\alpha3}\right)=0, \quad \alpha=1,2$$
$$R(u_1,u_2)=\tfrac{1}{2}(f-g)u_1^2+\tfrac{1}{2}(f+g)u_2^2-\tfrac{1}{8}\kappa_1(u_1^2+u_2^2)^2 \tag{1}$$
$$u_\alpha=\partial w_\alpha/\partial x, \quad \tau_{\alpha3}=\rho_0\nu\partial u_\alpha/\partial t\approx\nu\sqrt{\mu\rho_0}\,\partial u_\alpha/\partial x$$

Here w_α are the displacements of the particles, considered as a function of the Lagrangian coordinates x_1, x_2, $x_3\equiv x$, $\tau_{\alpha3}$ are the components of the viscous-stress tensor, ρ_0 is the density in the unstressed state, μ is the elastic Lamé coefficient, ν is the kinematic coefficient of viscosity, f and g are constants, where g $(g\ll f)$ is the anistropy parameter (small) and f is the characteristic velocity when there is no non-linearity and no anisotropy, and κ_1 is a constant with the dimensions of velocity, which characterizes non-linear effects.

The sign of the elastic constant κ_1 has a considerable influence on the behaviour of the simple waves and shock waves. System (1) contains two equations for the shear components of the strains, and the longitudinal components of the strains are expressed in terms of the shear components [4]

As already mentioned, for the same initial conditions $u_1=U_1$ and $u_2=U_2$ when $t=0$, $x>0$, and boundary conditions $u_1=u_1{}^*$, $u_2=u_2{}^*$ when $x=0$, $t>0$ over a certain range of the specified parameters there can be two different solutions consisting of a sequence of simple waves and shock waves, if the quantity $2g[(U_1^2+U_2^2)\kappa_1]^{-1}$ is fairly small [5, 6].

When $\kappa_1>0$ the solution of the first type contains a fast shock wave and a slow shock wave or a simple wave. The solution of the second type is a sequence of a complex fast wave and a slow shock or simple wave. The complex fast wave is a sequence of fast waves moving with very similar velocities (a fast shock wave with the Jouget condition behind it, a fast simple wave, and a fast shock wave with the Jouget condition in front of it).

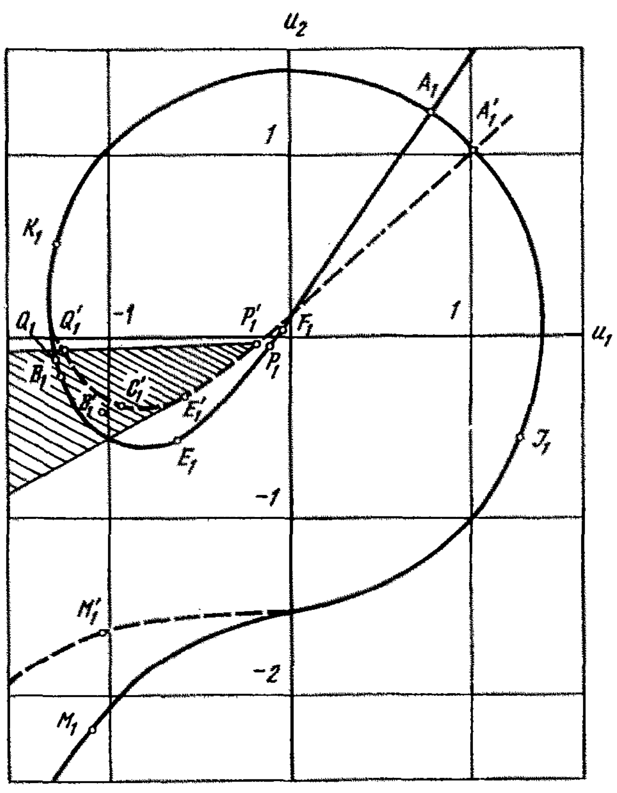

FIG. 1.

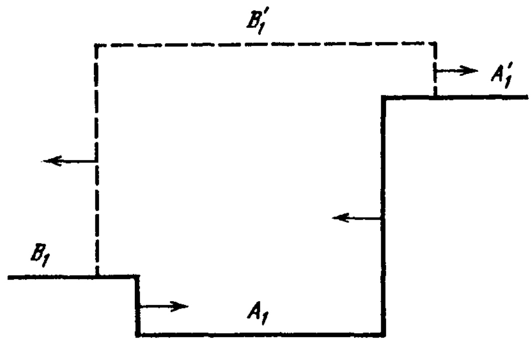

FIG. 2.

In the case when $\kappa_1 < 0$ the solution of the first type contains a fast shock wave or simple wave and a slow shock wave. The solution of the second type contains a fast shock wave or simple wave and a complex slow wave.

The problem of the interaction of shock waves moving in opposite directions has two solutions if the state in front and behind one of the systems of waves formed corresponds to non-uniqueness of the solution of the self-similar problem. Otherwise the solution can be obtained analytically [5–8].

1. Consider the following problem of the interaction of shock waves moving in opposite directions. Suppose $\kappa_1 > 0$. In Fig. 1 we show a shock adiabatic $A_1P_1E_1Q_1K_1A_1I_1M_1$ (the initial point A_1 (0.76; 1.22), $2g/\kappa_1 = 0.1$), constructed numerically.

Sections A_1I_1 and K_1E_1 are indicated on this shock adiabatic corresponding to a fast evolution shock wave, and the section A_1F_1, corresponding to a slow evolution shock wave.

In order to investigate the interaction that occurs between the oppositely travelling shock waves we formed a structure of a fast shock wave corresponding to the evolution jump from point A_1 (Fig. 1) to the point $A_1'(1;\ 1)$ and the structure of the shock wave moving in the opposite direction to it, corresponding to the evolution jump from the point A_1 to the point B_1 (−1.28; −0.2).

As a result of the interaction of the structures of these shock waves $A_1 \to A_1'$ and $A_1 \to B_1$ moving in opposite directions, a perturbation $A_1' \to B_1'$ will propagate to the right (Fig. 2) and a perturbation $B_1 \to B_1'$ to the left. The point B_1' is found from the condition that the difference in the values of the quantities u_1 and u_2 behind and in front of the perturbation $A_1' \to B_1'$ (Δu_1 and Δu_2) is equal to the difference between these quantities after and before the jump $A_1 \to B_1$. By what was said above we can assume that the jump $A_1 \to B_1$ after interaction with the jump $A_1 \to A_1'$ preserves its structure, but propagates in accordance with a new initial state of the medium corresponding to the point A_1'.

We will investigate the evolution of the perturbation $A_1' \to B_1'$. From the point A_1' as the initial point we draw a shock adiabatic $A_1'P_1'E_1'C_1'Q_1'A_1'M_1'$ (Fig. 1), represented by the dotted line. In Fig. 1 the region in which the solution of the problem is non-unique for the point A_1' as the initial point is shown hatched [1, 5]. The point B_1' belongs to this region of non-uniqueness.

To investigate the evolution of the perturbation $A_1' \to B_1'$ we formulated the following initial boundary-value problem for the system of equations (1). We specified as the initial conditions ($t = 0$) the structure of the jump $A_1 \to B_1$, propagating in accordance with the state A_1', and the right boundary condition ($t > 0$, $x = l$) was taken in the form $U_1 = U_2 = 1$ (the point A_1' in Fig. 1). The system of equations (1) was solved numerically as in [1, 2]. The numerical solution of the initial boundary-value problem in question for sufficiently long times is a sequence of a fast shock wave $A_1' \to C_1'$ and a slow shock wave $C_1' \to B_1'$, i.e. a self-similar asymptotic form of the first type.

After interaction between the waves $A_1 \to A_1'$ and $A_1 \to B_1$, the perturbation $B_1 \to B_1'$ will propagate towards the left. The point B_1' is not a point of the region in which the solution of the self-similar problem is non-unique for the initial point B_1. Hence the system of waves moving after interaction to the left is defined uniquely [5–8].

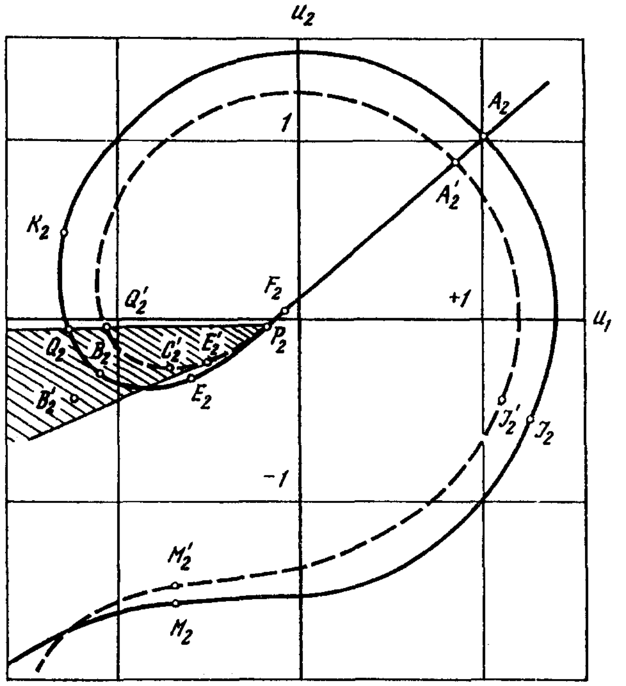

FIG. 3.

2. In the case when $\kappa_1 > 0$, $2g/\kappa_1 = 0.1$, we will consider the interaction of the shock waves $A_2 \to A_2'$ and $A_2 \to B_2$ (Fig. 3) (the coordinates of the points A_2, A_2' and B_2, respectively, are (1; 1), (0.86; 0.87) and (−1.1; −0.3)). In Fig. 3 we show the shock adiabatic $A_2F_2P_2E_2Q_2K_2A_2I_2M_2$, drawn from the point A_2 as the initial point, and the shock adiabatic $A_2'E_2'C_2'Q_2'A_2'I_2'M_2'$, drawn from the point A_2' as the initial point. As a result of the interaction between the oppositely travelling shock waves $A_2 \to A_2'$ and $A_2 \to B_2$, a perturbation $A_2' \to B_2'$ will propagate to the right, and a perturbation $B_2 \to B_2'$ will propagate to the left. The point B_2' belongs to the region in which the solution of the self-similar problem is non-unique (shown hatched in Fig. 3), corresponding to the point A_2' as the initial point.

We solved the initial boundary-value problem for system (1) numerically with the initial conditions ($t = 0$) corresponding to the structure of the jump $A_1 \to B_2$, propagating in accordance with the state A_2', with the right boundary condition ($t > 0$, $x = l$) $U_1 = 0.86$, $U_2 = 0.87$ corresponding to the point A_2' (Fig. 3), and with the left boundary condition ($t > 0$, $x = 0$) $u_1{}^* = -1.24$, $u_2{}^* = -0.43$, corresponding to the point B_2' (Fig. 3). Calculations showed that for fairly long times the solution consists of a fast shock wave $A_2' \to C_2'$ and a slow simple wave $C_2' \to B_2'$, i.e. we reach a self-similar asymptotic form of the first type.

The system of waves moving to the left after interaction between the oppositely travelling shock waves $A_2 \to A_2'$ and $A_2 \to B_2$ is defined uniquely, since the point B_2' does not belong to the region in which the solution of the self-similar problem is unique for the point B_2 as the initial point.

3. Suppose $\kappa_1 < 0$, $2g/\kappa_1 = -0.1$. In Fig. 4 we show the shock adiabatic $A_3F_3Q_3H_3A_3L_3B_3D_3$ (the initial point A_3 (1; 1), at which the sections of evolution A_3H_3 and L_3D_3, corresponding to the slow shock waves, and the section A_3N_3, corresponding to the fast evolution shock waves, are shown).

We will investigate the interaction between the oppositely travelling shock waves $A_3 \to A_3'$ and $A_3 \to B_3$ (Fig. 4) (the coordinates of the points A_3' and B_3 are, respectively, (1.2; 1.2) and (−0.72; −1.58)). As a result of the interaction between these two shock waves a perturbation $A_3' \to B_3'$ will propagate to the right and a perturbation $B_3 \to B_3'$ will propagate to the left. From the point A_3' as the initial point we draw a shock adiabatic $A_3'E_3'Q_3'H_3'A_3'L_3'D_3'$. In Fig. 4 the region in which the solution of the self-similar problem corresponding to the point A_3' as the initial point is non-unique is shown hatched. The point B_3' belongs to this region.

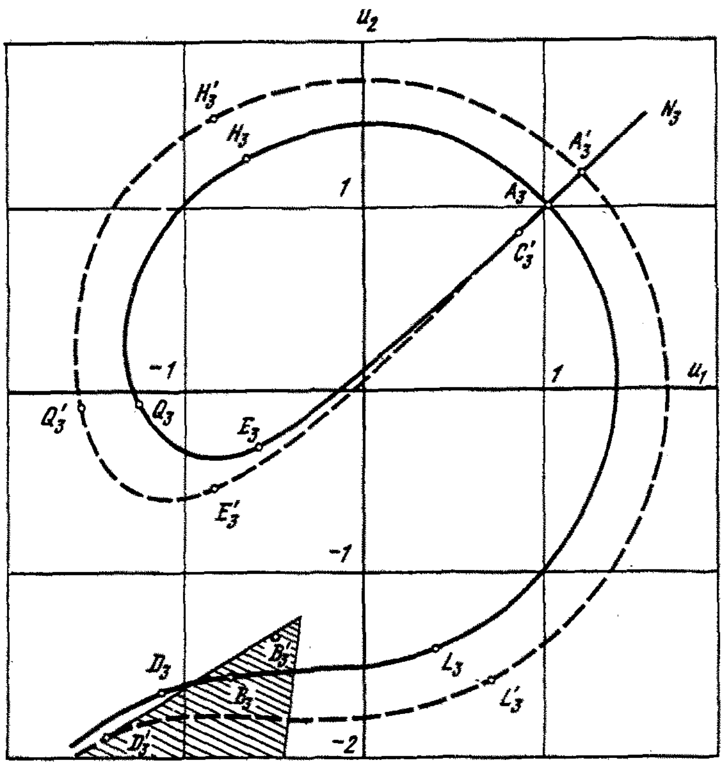

FIG. 4.

We solved the initial boundary-value problem for the system of equations (1) numerically with initial conditions ($t=0$) corresponding to the structure of the jump $A_3 \to B_3$, which propagates in accordance with the state A_3', i.e. for conditions on the right boundary ($t>0$, $x=l$) $U_1=U_2=1, 2$, corresponding to the point B_3' (Fig. 4), and with values on the left boundary ($t>0$, $x=0$) $u_1^*=-0.52$, $u_2^*=1.38$ corresponding to the point B_3'. Numerical calculations showed that the solution consists of a fast simple wave $A_3' \to C_3'$ and a slow shock wave $C_3' \to B_3'$, i.e. it is a self-similar asymptotic form of the first type.

The system of waves which travel to the left after interaction between the oppositely travelling shock waves $A_3 \to A_3'$ and $A_3 \to B_3$ is uniquely defined, since the point B_3' does not belong to the region in which the solution of the self-similar problem is non-unique for the point B_3 as the initial point.

4. Suppose $\kappa_1<0$, $2g/\kappa_1=-0.1$. In Fig. 5 we show the shock adiabatic $A_4B_4Q_4H_4A_4L_4B_4D_4$ (the initial point (A_4 (1; 1)), A_4H_4 and L_4D_4 are sections of evolution corresponding to the slow evolution shock wave, and A_4N_4 is an evolution section corresponding to the fast evolution shock wave.

As a result of the interaction between the shock waves $A_4 \to A_4'$ and $A_4 \to B_4$ (A_4 (0.76; 1.22), B_4 (−0.2; −1.53) in Fig. 5), travelling in opposite directions to one another, a perturbation $A_4' \to B_4'$ will propagate to the right, while a perturbation $B_4 \to B_4'$ will propagate to the left. $A_4'C_4'E_4'A_4'D_4'$ is a shock adiabatic drawn from the point A_4' as the initial point. In Fig. 5 the region in which the solution of the self-similar problem corresponding to the point A_4' as the initial point is non-unique is shown hatched. The point B_4' belongs to this region.

We solved the following initial boundary-value problem for the system of equations (1) numerically with initial conditions corresponding to the structure of the jump $A_4 \to B_4$, which propagates in accordance with the state A_4', i.e. for conditions on the right boundary ($t>0$, $x=l$) $U_1=0.76$, $U_2=1.22$ (the point A_4' in Fig. 5) and the conditions on the left boundary ($t>0$, $x=0$) $u^*_1=-0.44$, $u^*_2=1.31$ (the point B_4' in Fig. 5).

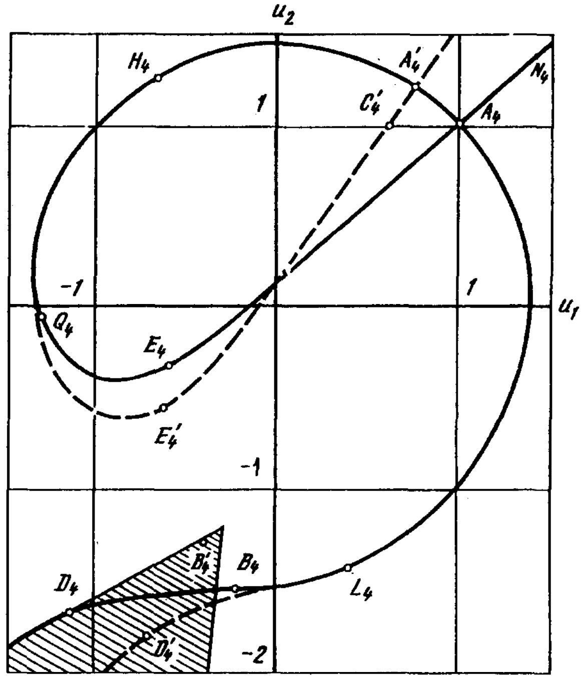

FIG. 5.

The solution consists of a fast simple wave $A_4' \to C_4'$ and a slow shock wave $C_4' \to B_4'$, i.e. it is a self-similar asymptotic form of the first type.

The system of waves moving to the left after interaction between the oppositely travelling shock waves $A_4 \to A_4'$ and $A_4 \to B_4$ is defined uniquely.

Hence, our investigation of the interaction between two shock waves of different types travelling in opposite directions has shown that in all the cases considered the solution reaches a self-similar asymptotic form corresponding to a solution of the first type of the self-similar problem of the change in the load on the boundary of a non-linearly elastic half-space. Similar results were obtained in [1, 2] when investigating problems of the interaction between two shock waves one of which overtakes the other.

The calculations carried out here, and also previously [1, 2], enable us to suggest the following. In those cases when, as a result of the interaction between the shock waves in an elastic medium with a negligibly small viscosity, problems arise regarding the decay of the initial discontinuity, their solutions at subsequent instants of time, corresponding to the asymptotic form, with respect to a viscosity approaching zero, are solutions of the first type. If we assume this hypothesis, then for a wide range of problems on waves in an elastic medium with a negligibly small viscosity, the proposed choice of solution will essentially be determined by the viscosity introduced into the problem.

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